PROBLEMS WITH ONE-SIDED CONSTRAINTS FOR NAVIER–STOKES EQUATIONS AND THE DYNAMIC CONTACT ANGLE

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The dynamic contact angle raises a problem because of incompatibility between the conditions at a free liquid surface and the attachment conditions at a solid wall in the region of the moving line of contact between three phases. That incompatibility was first pointed out in [1] and was demonstrated in [2] subject to minimal assumptions about the smoothness of the velocity pattern and the free surface. There are various ways of closing the formulation for the motion of a viscous incompressible liquid in the presence of a moving line (or point in the two-dimensional case) for a three-phase contact: replacing the attachment condition by the condition for slip on a certain part of the wall near the line of contact [3-6], an asymptotic approach in which the solution is not extended to the line of contact [7], and a suggestion that the dynamic contact angle should be taken as π when the liquid spreads over the dry surface and zero as it withdraws [2].

The first two methods require empirical information (the coefficients in the various forms of the slip condition and the inclination of the free surface to the wall at a small distance from it), while the third is not applicable to the motion for small values of the capillary number $Ca = \rho\nu V/\sigma$ (ρ is the density of the liquid, ν the kinematic viscosity, σ the surface tension, and V the normal velocity of the three-phase contact line with respect to the wall).

Here we propose a new approach to models for a viscous liquid having moving points of contact between the free boundary and a solid wall. We consider only two-dimensional stationary cases (planar and axisymmetric). The capillary number is taken as small, and the free boundary can be defined approximately from the capillary-equilibrium conditions, after which one has to solve the Navier–Stokes equations with mixed boundary conditions. The boundary to the flow region contains nodal points, which the type of boundary condition alters. In general, there is not even a general solution having a finite Dirichlet integral. The formulation thus needs to be modified, and the proposed modification consists in replacing the integral identity satisfied by the general solution (if it exists) by a variational inequality. The one-sided constraint involved in the new formulation consists in sign-definiteness in the tangential component of the velocity on part of the boundary.

It is shown that this inequality is soluble, and that the solution is unique for linearized Navier–Stokes equations. Particular cases are considered of capillary filling and flow in a rotating container, where the solution has a natural physical interpretation.

1. Capillary Filling Model. Let a viscous incompressible liquid fill a planar vertical capillary of width 2a under gravity g. In a coordinate system linked to the moving points of contact, the flow is of stationary character, and we transfer to that system. We assume also that the motion above the points of contact tends to Poiseuille type, on which is superimposed a constant flow with velocity −V, which is defined by the condition for zero flow rate through the cross section: $V = ga^2/3\nu$. We introduce dimensionless variables, where the scales for length, velocity, and pressure are a, V, and $\rho\nu Va$ correspondingly. Then the equations of motion are

$$\Delta \mathbf{v} - \nabla p + 3\mathbf{e}_1 = \mathrm{Re}\,\mathbf{v}\cdot\nabla\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0. \tag{1.1}$$

Here $\mathbf{v} = (v_1, v_2)$ and p are the dimensionless velocity vector and the pressure, with $\mathbf{e}_1 = (1, 0)$; $\mathrm{Re} = Va/\nu$ the Reynolds number.

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The flow is taken as asymmetrical with respect to the axis of the capillary $x_2 = 0$, and as the physical formulation is invariant under displacement along the x_1 axis, it follows that the points of contact can be taken as $x_1 = 0$, $x_2 = -1$ and $x_1 = 0$, $x_2 = 1$.. The solution to (1.1) has to be found in the region Ω (Fig. 1) bounded by the walls of the capillary $x_1 < 0$, $x_2 = -1$ and $x_1 < 0$, $x_2 = 1$ and the free boundary $\Gamma = \{x_1, x_2 : x_1 = f(x_2),\ |x_2| \leqslant 1\}$, which is not known in advance and has to be determined along with $\mathbf{v}$, p. The traditional formulation of the boundary-value problem for this system lies in specifying the attachment conditions at the capillary walls:

$$\mathbf{v} = -\mathbf{e}_1,\ x_1 < 0,\ x_2 = \pm 1, \tag{1.2}$$

together with the kinematic condition

$$\mathbf{v} \cdot \mathbf{n} = 0,\ (x_1,\ x_2) \in \Gamma \tag{1.3}$$

and the dynamic conditions

$$\mathbf{s} \cdot D \cdot \mathbf{n} = 0,\ (x_1,\ x_2) \in \Gamma; \tag{1.4}$$

$$H = K + \mathrm{Ca}(p - 2\mathbf{n} \cdot D \cdot \mathbf{n}),\ (x_1,\ x_2) \in \Gamma \tag{1.5}$$

at the free boundary [8]. Here the symbols are as follows: $\mathbf{n}$ and $\mathbf{s}$ unit vectors for the exterior normal and tangent to the $\mathbf{v}$ curve, $D = D(\mathbf{v})$ the deformation-rate tensor corresponding to vector $\mathbf{v}$, with elements $D_{ij} = 0.5(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$, $i,\ j = 1,\ 2$; H the curvature of curve Γ; and $K = -ap_a/\sigma$ a constant (p_a is atmospheric pressure); $\mathrm{Ca} = \rho\nu V/a$. . It is assumed that $H > 0$ if Γ is convex outwards from the liquid.

Condition (1.3) means that Γ is a flow line. By virtue of (1.4), the tangential stress is zero on Γ. Condition (1.5) expresses the fact that the difference between the capillary pressure and the normal stress at points on the free boundary is equal to the atmospheric pressure.

As Ω is noncompact, we have to impose a certain condition at $x_1 \to -\infty$ in order to solve (1.1). We assume that far from the free boundary we get a flow close to a superposition of a Poiseuille flow and a uniform one:

$$v_1 \to (1 - 3x_2^2)/2,\ v_2 \to 0,\ p \to \mathrm{const},\ x_1 \to -\infty,\ |x_2| \leqslant 1. \tag{1.6}$$

Finally, one needs to impose an additional condition at the points of contact between the free boundary and the solid walls, which we take by analogy with capillary equilibrium for a liquid [3] as

$$f' = \mp \operatorname{ctg} \gamma \quad \text{at} \quad x_2 = \pm 1 \tag{1.7}$$

with $(\gamma \in (0,\ \pi]$ a constant quantity that is identified with the dynamic contact angle [3].

2. Asymptotic Simplification. The (1.1)-(1.7) treatment is one with an unknown boundary for a nonlinear equation system in an unbounded region. Also, if $\gamma < \pi$ in (1.7), the kinetic-energy dissipation rate near the points of contact is represented by a divergent integral [2], which casts doubt on the physical significance of the solution even if it exists in a wider class of functions than that usually employed in solving boundary-value problems for Navier–Stokes equations [9]. We reduce (1.1)-(1.7) to a more manageable treatment on the assumption that the defining parameters Re and Ca are small, and that their dependence on the input data is $\mathrm{Re} = ga^3/3\nu^2$, $\mathrm{Ca} = \rho g a^2/3\sigma$ (the latter formula enables one to identify the capillary number with the Bond number here). One can make Re and Ca small by making the half-width of the channel a small with fixed values for the other input data. We put Re = 0 in (1.1) to get a Stokes system:

$$\Delta \mathbf{v} - \nabla p + 3\mathbf{e}_1 = 0,\ \nabla \cdot \mathbf{v} = 0. \tag{2.1}$$

The transition to the limit on the capillary number in (1.1)-(1.7) is based on the assumption that $\mathbf{v} \to 0$, $\nabla \mathbf{v} \to 0$, $\Delta \mathbf{v} \to 0$, when $\mathrm{Ca} \to 0$, but then $p\mathrm{Ca} \to K_0 = \mathrm{const}$, which corresponds to the limiting state of capillary equilibrium in the absence of external forces (see [10] on the procedure for expanding the solution to a problem with a free boundary with respect to a parameter responsible for the deformability of the free surface, that parameter in our case being Ca). From (1.5) in the limit Ca → 0 we get $H = K + K_0 = \text{const}$, so in the principal order with respect to Ca, curve Γ is an arc of a circle:

$$(x_1 + \operatorname{tg}\gamma)^2 + x_2^2 = (\cos\gamma)^{-2}, \ |x_2| \leqslant 1. \tag{2.2}$$

The circle's parameters are determined in accordance with (1.7).

We make a further simplification partially justified by the St. Venant principle for a Stokes system [11]: we assume that for $x_1 \to -\infty$, the solution tends rapidly to a Poiseuille one, and we transfer condition (1.6) from infinity to a finite distance:

$$v_1 = (1 - 3x_2^2)/2, \ v_2 = 0, \ x_2 = -l, \ |x_2| \leqslant 1. \tag{2.3}$$

Here ℓ is a sufficiently large positive constant (in any case, $\ell > 1$). Then the simplified treatment for the capillary filling will be examined below.

We have to find the solution $\mathbf{v}, p$ to system (2.1) in the region $\Pi = \{x_1, x_2\colon\ 0 < x_2 < 1, -l < x_1 < [(\cos\gamma)^{-2} - x_2^2]^{1/2} - \operatorname{tg}\gamma\}$ (hatched in Fig. 1) that satisfies (2.3) and

$$\mathbf{v} = \mathbf{e}_1, \ -l \leqslant x_1 \leqslant 0, \ x_2 = 1; \tag{2.4}$$

$$\mathbf{v}\cdot\mathbf{n} = 0, \ (x_1, x_2) \in \Gamma, \ x_2 \geqslant 0; \tag{2.5}$$

$$\mathbf{s}\cdot D(\mathbf{v})\cdot\mathbf{n} = 0, \ (x_1, x_2) \in \Gamma, \ x_2 \geqslant 0; \tag{2.6}$$

$$v_2 = 0, \ \partial v_1/\partial x_2 = 0, \ -l \leqslant x_1 \leqslant \operatorname{tg}(\gamma/2 - \pi/4), \ x_2 = 0, \tag{2.7}$$

in which **n** and **s** are unit vectors for the normal and tangent to Γ as defined by (2.2).

The (2.1)-(2.7) treatment is much simpler than the initial one: it is linear and the definition region for the solution is bounded. There are nodal points on the boundary of region Π. Three of them ($x_1 = -l, x_2 = 1;\ x_1 = -l,\ x_2 = 0;\ x_1 = \operatorname{tg}(\gamma/2 - \pi/4),\ x_2 = 0$), are right-angle vertices, and the boundary conditions on the lines forming those angles are consistent, so no singularities arise in the solution at those points (this statement is made without proof, but there is no doubt that it is correct, since those nodal points are of artificial origin). As regards the points of contact between the free boundary and the solid wall $x_1 = 0, x_2 = 1$, here the situation is different. If the contact angle satisfies $0 < \gamma < \pi$, there is no solenoidal vector field **v** that satisfies (2.4) and (2.5) and which has a finite Dirichlet integral [2]:

$$\int_\Pi \nabla\mathbf{v} : \nabla\mathbf{v}\, dx < \infty$$

($\nabla\mathbf{v}$ is the gradient in vector **v** and the colon represents tensor convolution).

We leave aside the question whether (2.1)-(2.7) is soluble in the class of functions **v** having unbounded Dirichlet integrals; such solutions describe flows having infinite energy dissipation rate, so they are only of academic interest.

The values $\gamma = 0$ and $\gamma = \pi$ are exceptional. We do not consider $\gamma = 0$ because it is physically unrealizable in the capillary-filling case. With $\gamma = \pi$, we follow the [12, 13] method to demonstrate existence and uniqueness for the general solution to (2.1) -(2.7) with finite Dirichlet integral. The solubility is related to the continuity in the velocity pattern near the points of contact, which occurs for $\gamma = \pi$ when the boundary of region Π at $x_1 = 0,\ x_2 = 1$ is smooth.

This observation suggests that we should shift the point of change in conditions (2.4) with (2.5) and (2.6) along the solid wall in such a way that it is not a nodal point on the boundary. We thus assume that the attachment condition is replaced by the following conditions on a certain part of the wall adjoining the point of contact:

$$v_2 = 0, \ -\delta \leqslant x_1 \leqslant 0, \ x_2 = 1; \tag{2.8}$$

$$\partial v_1/\partial x_2 = 0, \ -\delta < x_1 \leqslant 0, \ x_2 = 1 \quad (2.9)$$

($\delta < \ell$ is a small positive quantity). On the rest of the wall, we retain the attachment condition:

$$\mathbf{v} = -\mathbf{e}_1, \ -l \leqslant x_1 < -\delta, \ x_2 = 1. \quad (2.10)$$

Condition (2.8) is that for no flow, while (2.9) is naturally called the ideal sliding condition, which is the simplest of the conditions of that kind proposed in [3-6] and is used to describe the motion of the three-phase contact line along a rough wall (see the discussion on this in [14]).

The small parameter $\delta > 0$ can be taken as the regularization parameter because (2.1), (2.3), (2.5)-(2.10) has a general solution, which is unique, for any $\gamma \in (0, \pi]$. That assertion is essentially a consequence of the [12, 13] results. Also, a variational formulation is possible: a solution can be obtained by minimizing the energy-dissipation functional

$$J(\mathbf{v}) = 2\int_{\Pi} D(\mathbf{v}):D(\mathbf{v})\,dx \quad (2.11)$$

on the set of solenoidal vector functions $\mathbf{v}$ having components v_i $(i = 1, 2)$ from the Sobolev class $H^1(\Pi)$, that satisfy (2.3), (2.5), (2.8), (2.10) and the first condition in (2.7), which is the principal one for the (2.11) functional. Conditions (2.6) and (2.9) with the second one from (2.7) are natural ones, and they are met at the turning points in J.

3. Variational Inequality. The Sec. 2 regularization of (2.1)-(2.7) contains δ, which cannot be determined from classical concepts on viscous-liquid dynamics. Here we present a modified formulation not containing additional parameters, which is also based on the variational principle associated with (2.11), but instead of (2.8) and (2.10), which appear in the principal conditions for the variational treatment, we specify obedience to

$$v_2 = 0, \ -l \leqslant x_1 \leqslant 0, \ x_2 = 1 \quad (3.1)$$

and

$$v_1 \leqslant -1, \ -l \leqslant x_1 \leqslant 0, \ x_2 = 1. \quad (3.2)$$

Then (2.5) and (3.1) with the first equation in (2.7) show that the no-flow condition is met on the entire boundary of region Π apart from the segment $x_1 = -l, \ 0 \leqslant x_2 \leqslant 1$. The condition expressed by (3.2) is basic for Sec. 3. At the points where $v_1 = -1$, (3.2) becomes the attachment condition. At the same time, (3.2) does not rule out the possibility that the liquid slides with respect to the wall. For $\gamma < \pi$ and finite J, that certainly occurs for small $|x_1|, x_1 < 0$ (see Sec. 2).

The assumption $v_1 \leqslant -1$ is a formalization of the intuitive view that the wall retards the motion, the reason for which is the force of gravity, it means that at points where the liquid moves relative to the wall, the absolute value of v_1 cannot be less than the wall velocity.

It is inconvenient to minimize (2.11) because one of the principal conditions, (2.3), is inhomogeneous. We make the substitution

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \quad (3.3)$$

in which $\mathbf{w}$ is a vector having components

$$w_1 = \partial(\zeta\Psi)/\partial x_2, \ w_2 = -\partial(\zeta\Psi)/\partial x_1. \quad (3.4)$$

Here $\Psi(x_2) = (x_2 - x_2^3)/2$ is the current function corresponding to Poiseuille-type flow, while $\zeta(x_1, x_2)$ is a truncating function. If $\gamma \geqslant \pi/2$ (which is of the main physical interest), that function can be taken as $\zeta = \eta(x_1)$, in which the function $\eta \in C^\infty[-l, 0]$ is subject to the conditions $\eta = 1$ for $-l \leqslant x_1 \leqslant -\varepsilon, \eta = 0$ for $-\varepsilon/2 \leqslant x_1 \leqslant 0$ and $0 \leqslant \eta \leqslant 1, \ d\eta/dx_1 \leqslant 0$

for all $x_1 \in [-l, 0]$, but otherwise is arbitrary. For $0 < \gamma < \pi/2$ we put $\zeta = \eta\langle l^{-1}x_1\{l - [(\cos\ \gamma)^{-2} - x_2^2]^{1/2} + \mathrm{tg}\gamma\}\rangle$. . It is clear that in both cases $\zeta \in C^\infty(\bar{\Pi})$ and that $\zeta = 0$ near the part Γ of the boundary to region Π. We further assume $\gamma \geq \pi/2$, which does not produce any substantial simplifications but reduces the exposition somewhat.

The functional I is defined by

$$I(\mathbf{u}) = 2\int_\Pi [D(\mathbf{u}) : D(\mathbf{u}) + 2D(\mathbf{u}) : D(\mathbf{w})]\, dx. \tag{3.5}$$

Minimizing I is equivalent to that for J, since $J(\mathbf{u} + \mathbf{w}) - I(\mathbf{u})$ is independent of $\mathbf{u}$. . We put: $\Sigma_1 = \{x_1, x_2: -l \leqslant x_1 \leqslant 0, x_2 = 1\}$, $\Sigma_2 = \{x_1, x_2: x_1 = -l, 0 \leqslant x_2 \leqslant 1\}$, $\Sigma_3 = \{x_1, x_2: -l \leqslant x_1 \leqslant \mathrm{tg}(\gamma/2 - \pi/4), x_2 = 0\}$. The union of the sets Γ, Σ_1, Σ_2 and Σ_3 forms the boundary of region II. We define the functional space $\mathbf{H}^1(\bar{\Pi})$ as a closure in norm

$$\|\boldsymbol{\varphi}\|_{\mathbf{H}^1(\Pi)} = \left[\int_\Pi D(\boldsymbol{\varphi}) : D(\boldsymbol{\varphi})\, dx\right]^{1/2}$$

of the set of smooth vector functions $\boldsymbol{\varphi}$ solenoidal in Π that satisfy the conditions

$$\boldsymbol{\varphi}\cdot\mathbf{n} = 0,\ (x_1, x_2) \in \Gamma; \tag{3.6}$$

$$\varphi_2 = 0,\ (x_1, x_2) \in \Sigma_1; \tag{3.7}$$

$$\boldsymbol{\varphi} = 0,\ (x_1, x_2) \in \Sigma_2; \tag{3.8}$$

$$\varphi_2 = 0,\ (x_1, x_2) \in \Sigma_3. \tag{3.9}$$

Space $\mathbf{H}^1(\Pi)$ becomes a Hilbert one if for any pair of its elements $\boldsymbol{\varphi}$ and ψ we define the scalar product from the formula

$$(\boldsymbol{\varphi}, \boldsymbol{\psi})_{\mathbf{H}^1(\Pi)} \equiv a(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_\Pi D(\boldsymbol{\varphi}) : D(\boldsymbol{\psi})\, dx.$$

The Korn inequality [13] applies for functions from space $\mathbf{H}^1(\Pi)$

$$\int_\Pi \nabla\boldsymbol{\varphi} : \nabla\boldsymbol{\psi}\, dx \leqslant C_1 \|\boldsymbol{\varphi}\|^2_{\mathbf{H}^1(\Pi)}$$

together with the Poincaré–Friedricks inequality [9]

$$\int_\Pi |\boldsymbol{\varphi}|^2\, dx \leqslant C_2 \|\boldsymbol{\varphi}\|^2_{\mathbf{H}^1(\Pi)}$$

with positive constants C_1 and C_2 independent of φ . Finally, by K we denote the set $K = \{\boldsymbol{\varphi} \in \mathbf{H}^1(\Pi): \varphi_1 \leqslant \eta(x_1) - 1 \text{ on } \Sigma_1\}$, which is convex and closed in $\mathbf{H}^1(\Pi)$. . (By virtue of the Koran and Poincaré–Friedricks inequalities, the components φ_1 and φ_2 of vector $\boldsymbol{\varphi} \in \mathbf{H}^1(\Pi)$ belong to the Sobolev space $H^1(\Pi)$. , and according to the embedding theorem, the trace of φ_1 on the segment Σ_1 is a function from the class $H^{1/2}(\Sigma_1)$. , so the statement «$\varphi_1 \leqslant \eta(x_1)$ – almost everywhere in Σ_1 » is meaningful).

Variational-inequality theory [15] implies that minimizing (3.5) on the set $K \subset \mathbf{H}^1(\Pi)$ is equivalent to solving the variational inequality

$$a(\mathbf{u}, \mathbf{u} - \boldsymbol{\varphi}) \leqslant L(\mathbf{u} - \boldsymbol{\varphi})\ \forall\boldsymbol{\varphi} \in K, \tag{3.10}$$

in which L is a linear functional upon the space $H^1(\Pi)$. defined by

$$L(\mathbf{u}) = -\int_\Pi D(\mathbf{w}) : D(\mathbf{u})\, dx \tag{3.11}$$

(vector $\mathbf{w}$ is defined by (3.4)). The Lyons–Stampacci theorem [15] implies that a solution exists to (3.10), which is unique.

It is readily shown that (2.1) is satisfied by the vector function v constructed from u by means of (3.3) along with a suitable scalar function p in region Π. In fact, let $\boldsymbol{\psi} = (\psi_1, \psi_2)$, in which ψ_1 and ψ_2 are any functions from $C_0^\infty(\Pi)$ such that $\nabla\cdot\boldsymbol{\psi} = 0$. Then the vector functions $\boldsymbol{\varphi}_1 = \mathbf{u} - \boldsymbol{\psi}$ and $\boldsymbol{\varphi}_2 = \mathbf{u} + \boldsymbol{\psi}$ belong to K. In (3.10) we put in turn $\boldsymbol{\varphi}_1 = \mathbf{u} - \boldsymbol{\psi}$ and $\boldsymbol{\varphi}_2 = \mathbf{u} + \boldsymbol{\psi}$, to get $a(\mathbf{u}, \boldsymbol{\psi}) \leqslant L(\boldsymbol{\psi})$, $a(\mathbf{u}, -\boldsymbol{\psi}) \leqslant L(-\boldsymbol{\psi})$, which means $a(\mathbf{u}, \boldsymbol{\psi}) = L(\boldsymbol{\psi})$. We use the definition (3.11) of functional L and apply Green's formula to the Stokes system [9] to get

$$\int_\Pi \Delta\mathbf{v}\cdot\boldsymbol{\psi}\, dx = 0.$$

As the solenoidal vector $\boldsymbol{\psi} \in \mathbf{C}_0^\infty(\Pi)$ is arbitrary, the latter equation means that there exists a single-valued function p such that the first of the (2.1) equations is met in Π in the distribution sense.

We now interpret the boundary conditions. Here we additionally assume that the solution u to (3.10) is continuous in Π. According to (3.8), for $\mathbf{u} \in K$ we have u = 0 on Σ_2, so v = u + w satisfies (2.3). Further, we consider an arbitrary function $\boldsymbol{\varphi} \in K \cap \overset{\circ}{\mathbf{H}}{}^1(C_{\xi,\varepsilon_1})$, in which $C_{\xi,\varepsilon_1} = \{x_1, x_2\colon (x_1 - \xi)^2 + x_2^2 < \varepsilon_1^2\}$ and $\boldsymbol{\varphi} = 0$ for $(x_1, x_2) \in \Pi \setminus C_{\xi,\varepsilon_1} \cap \Pi$. Here (ξ, 0) is any internal point on the segment Σ_3 and $\varepsilon_1 = 0.5 \min(1, l + \xi, \operatorname{tg}(\gamma/2 - \pi/4) - \xi)$. As $\mathbf{u} \in \mathbf{H}^1(\Pi)$, we can transform $a(\mathbf{u}, \mathbf{u} - \boldsymbol{\varphi}) - L(\mathbf{u} - \boldsymbol{\varphi})$ by means of Green's formula. We use (2.1) and (3.9) to get from (3.10) that

$$\int_{\xi-\varepsilon_1}^{\xi+\varepsilon_1} \frac{\partial v_1}{\partial x_2}(x_1, 0)\,[u_1(x_1, 0) - \varphi_1(x_1, 0)]\, dx_1 \geqslant 0.$$

That $\boldsymbol{\varphi}$ belongs to set K imposes no constraints on the values of φ_1 at points on Σ_3, but then the latter inequality implies that the second condition in (2.7) is met.

Consequently, the condition is a natural one for functional (3.5). An analogous argument shows that (2.6) is obeyed for points on Γ for v = u + w.

These properties of (2.5)-(2.7) of v can then be used again with Green's formula to transform (3.10), which gives

$$\int_{-l}^{0} \frac{\partial v_1}{\partial x_2}(x_1, 1)\,[u_1(x_1, 1) - \varphi_1(x_1, 1)]\, dx_1 \leqslant 0. \tag{3.12}$$

Let Σ' be any closed subinterval in segment Σ_1 and μ a function in the $C_0^\infty(\Sigma')$ class satisfying $0 \leqslant \mu \leqslant 1$, but otherwise arbitrary. We consider the vector function $\boldsymbol{\varphi} \in \mathbf{H}^1(\Pi)$, for which the trace of the first component in Σ_1 is given by $\varphi_1|_{\Sigma_1} = (1 - \mu)u_1|_{\Sigma_1} + \mu(\eta - 1)$ (there is no doubt that such $\boldsymbol{\varphi}$ exists). For $\mathbf{u} \in K$, we have $\varphi_1(x_1, 1) \leqslant \eta(x_1) - 1$ for all $x_1 \in [-l, 0]$ so $\boldsymbol{\varphi} \in K$. We substitute that expression for $\varphi_1(x_1, 1)$ into (3.12) to get

$$\int_{\Sigma'} \frac{\partial v_1}{\partial x_2}(u_1 - \eta + 1)\,\mu\, d\Sigma' \leqslant 0.$$

We repeat a similar argument, in which $\varphi_1|_{\Sigma_1} = (1 + \mu)u_1|_{\Sigma_1} - \mu(\eta - 1)$, to get from (3.12) that

$$\int_{\Sigma'} \frac{\partial v_1}{\partial x_2}(u_1 - \eta + 1)\,\mu\, d\Sigma' \geqslant 0.$$

Consequently

$$\int_{\Sigma'} \frac{\partial v_1}{\partial x_2}(u_1 - \eta + 1)\mu\, d\Sigma' = 0.$$

As $\Sigma' \subset \Sigma_1$ is arbitrary, and as the nonnegative function μ from $C_0^\infty(\Sigma')$ is arbitrary, we recall that $u_1 - \eta = v_1$ on Σ_1 by virtue of (3.3) and (3.4), so we conclude that

$$(v_1 + 1)\partial v_1/\partial x_2 = 0 \quad \text{on } \Sigma_1. \tag{3.13}$$

We now consider the function $\boldsymbol{\varphi} \in \mathbf{H}^1(\Pi)$, for which $\varphi_1|_{\Sigma_1} = u_1|_{\Sigma_1} - \chi$. Here χ is any functions on Σ_1 that satisfies $\chi \in H^{1/2}(\Sigma_1) \cap C^0(\Sigma_1), \chi \geqslant 0$. If $\mathbf{u} \in K$, it is clear that $\boldsymbol{\varphi} \in K$, and we return to (3.12) to get

$$\int_{\Sigma'} \frac{\partial v_1}{\partial x_2} \chi\, d\Sigma_1 \leqslant 0.$$

As $\chi \geqslant 0$ is arbitrary, we have

$$\partial v_1/\partial x_2 \leqslant 0 \quad \text{on } \Sigma_1, \tag{3.14}$$

to be understood in the sense of $H^{-1/2}(\Sigma_1)$.

Then (3.2) with (3.14) and (3.1) with (3.13) represent a set of conditions satisfied by the solution to that minimization on part Σ_1 of the boundary of Π. Continuity of v_1 in Π implies that one can consider the set $\Sigma_1^- = \{(x_1, x_2) \in \Sigma_1 : v_1(x_1, x_2) + 1 < 0\}$. From (3.2) and (3.12)-(3.14) we proceed as in examination of the Signorini problem [15] to show that

$$\partial v_1/\partial x_2 = 0 \quad \text{on } \Sigma_1^- \tag{3.15}$$

in the sense of a measure on Σ_1^-. (3.15) means that on the set Σ_1^-, where the attachment condition is not met (since $v_1 < -1$ on that set), one instead satisfies the condition for ideal slip. On the other hand, (3.13) shows that at those points in the $\Sigma_1 \setminus \Sigma_1^-$ set where $\partial v_1/\partial x_2 < 0$, the attachment condition is necessarily met.

Unfortunately, we have no information on the structure of set Σ_1^-. Arguments based on the [1, 2] results, which are briefly presented in sec. 2, lead to the inclusion that for $0 < \gamma < \pi$ the Σ_1^- set is nonempty and contains a certain interval $(-\varepsilon_2, 0)$ analogous to the suction gap in the filtration of a liquid through a dam [15].

To conclude sec. 3, we note that conditions (2.7), which are satisfied by the solution $(\mathbf{v}, p)$ as constructed with the one-sided constraint for the (2.1) system, enable one to extend that solution into the region $\tilde{\Pi}$ symmetrical in relation to Π about the x_1 axis.

That is, v_1 and p are continued in an even fashion in $\tilde{\Pi}$, while v_2 is continued in an odd fashion with respect to variable x_2. The continuation is defined in the region $(x_1, x_2) \in \Omega$, $x_1 \geqslant -l$, and it may be called an approximate solution to the modified treatment for filling of a planar capillary (asymptotic for $\mathrm{Re} \to 0$, $\mathrm{Ca} \to 0$). In that solution, the energy dissipation rate is finite, and the dynamic contact angle takes any fixed value in the range $(0, \pi)$. An important feature is that the tangential stress at the wall as expressed by (3.14) is sign-definite. The correct sign in that inequality is an additional confirmation of the physical reasonableness of the initial (3.2) assumption.

4. Rotating Container. We consider a two-dimensional treatment for the stationary motion of a liquid partially filling a circular region with radius R. The boundary to the region is a solid impermeable wall rotating with constant angular velocity $\omega > 0$ around the center of the circle. The liquid is acted on by the force of gravity (acceleration g), which is directed along the negative x_2 axis. We take the origin at the center of the circle and denote the region occupied by the liquid by Ω, as in Sec. 1. Part of the boundary of Ω is free and is denoted by Γ (Fig. 2). The other part of the boundary, Σ, is an arc of the circle $r \equiv (x_1^2 + x_2^2)^{1/2} = 1$, and on it we specify obedience to the low-flow condition and an inequality that resembles (3.2). The scales are: length R, velocity ωR, and pressure $\rho\omega\nu$.

This case has not only a dynamic contact angle γ $(0 < \gamma < \pi)$ and parameter α $(|\alpha| < \pi/2)$, which expresses the container filling (Fig. 2), but also three dimensionless positive parameters: the Reynolds number $\mathrm{Re} = \omega^2 R/\nu$, the Bond number $\mathrm{Bo} = \rho g R^2/\sigma$, and the capillary number $\mathrm{Ca} = \rho\nu\omega R/\sigma$.

As before, we take Ca as small, whereas no such constraint is imposed on Re and Bo. One can thus act in the spirit of Sec. 2 to reduce the treatment involving an unknown boundary approximately to one for a fixed region. We assume that Γ line is unambiguously projected on the x_1 axis and define the equation as $x_2 = f(x_1)$. In the first approximation with respect to the small Ca, we get a boundary-value treatment for f:

$$\left(\frac{f'}{\sqrt{1+f'^2}}\right)' - \mathrm{Bo}\, f = C \quad \text{for} \quad |x_1| < \cos\alpha; \tag{4.1}$$

$$C = \frac{\cos(\gamma-\alpha)}{\cos\alpha} - \frac{\mathrm{Bo}}{2\cos\alpha}\int\limits_{-\cos\alpha}^{\cos\alpha} f(x_1)\,dx_1; \tag{4.2}$$

$$f' = \pm\mathrm{ctg}\,(\gamma-\alpha) \quad \text{for} \quad x_1 = \pm\cos\alpha \tag{4.3}$$

(a prime denotes differentiation with respect to x_1). The solutions describe the forms of equilibrium for a heavy liquid in a circular region having the unambiguous projection property (in [16], equilibrium forms were considered that are not so projected on the x_1 axis). The (4.1)-(4.3) treatment is soluble subject to certain constraints of inequality type on α, γ, and Bo (in addition to those above). The solubility conditions are contained in inexplicit form in [15]. We assume that those conditions are met. There is a solution in the form of an even function $f(x_1)$ for any Bo > 0 and α close to γ − π/2. If α = γ − π/2, the unique solution is f = sin α.

We now define the region $\Omega \subset \mathbb{R}^2$ by $r < 1$, $x_2 < f(x_1)$, with the boundary $\partial\Omega = \bar\Gamma \cup \bar\Sigma$, where $\Gamma = \{x_1,\ x_2\colon\ x_2 = f(x_1),\ |x_1| < \cos\alpha\}$, $\Sigma = \{x_1,\ x_2\colon\ r = 1,\ x_2 < \sin\alpha\}$. The points $x_1 = -\cos\alpha$, $x_2 = \sin\alpha$ and $x_1 = \cos\alpha$, $x_2 = \sin\alpha$ are ones of three-phase dynamic contact.

The formulation consists in determining the pair of functions **v** and p satisfying the Navier–Stokes equations

$$\Delta\mathbf{v} - \nabla p = \mathrm{Re}\ \mathbf{v}\cdot\nabla\mathbf{v},\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega \tag{4.4}$$

and the conditions at the boundary of the region

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma; \tag{4.5}$$

$$\mathbf{s}\cdot D(\mathbf{v})\cdot\mathbf{n} = 0 \quad \text{on } \Gamma; \tag{4.6}$$

$$v_r = 0,\ v_\theta = 1 \quad \text{on } \Sigma_2; \tag{4.7}$$

$$v_r = 0 \quad \text{on } \Sigma_1 \cup \Sigma_3; \tag{4.8}$$

$$v_\theta \leqslant 1 \quad \text{on } \Sigma_1 \cup \Sigma_3. \tag{4.9}$$

Here p is the difference between the pressure in the liquid and the hydrostatic pressure in the liquid and the hydrostatic pressure $gRx_2/\nu\omega$; , while **n** and s are unit vectors for the exterior normal and the tangent to Γ; v_r and v_θ are the projections of vector v on the axes of the polar coordinate system r and $\theta = \mathrm{arctg}\,(x_2/x_1)$. . By Σ_i $(i = 1, 2, 3)$, we denote the componets of set Σ defined by $\Sigma_1 = \{(x_1,\ x_2) \in \Sigma,\ -\pi - \alpha < \theta < -\pi - \alpha + \delta\}$, $\Sigma_2 = \{(x_1,\ x_2) \in \Sigma, -\pi - \alpha + \delta < \theta < \alpha - \delta\}$, $\Sigma_3 = \{(x_1,\ x_2) \in \Sigma,\ \alpha - \delta < \theta < \alpha\}$ (Fig. 2) $\delta \in (0,\ \pi/2 + \alpha)$ being a constant.

Inequality (4.9) means that the absolute value of the tangential velocity of the liquid on the parts of the wall adjoining the free boundary does not exceed the velocity of the wall itself. This reflects the rational concept that the free boundary and the force of gravity have retarding effects on this flow as generated by the container's rotation (in contrast, in the capillary filling, the liquid is retarded at the walls, so the corresponding (3.2)

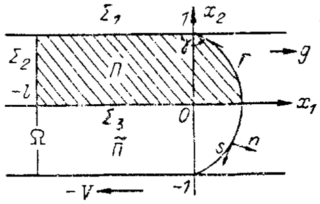

Fig. 1

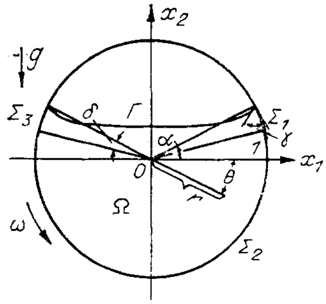

Fig. 2

rewritten in terms of $|v_1|$ will have a sign different from that in (4.9)).

To close the formulation, we supplement (4.4)-(4.9) with

$$\partial v_\theta/\partial r - v_\theta \leqslant 0 \quad \text{on } \Sigma_1 \cup \Sigma_3; \tag{4.10}$$

$$(v_\theta - 1)(\partial v_\theta/\partial r - v_\theta) = 0 \quad \text{on } \Sigma_1 \cup \Sigma_3. \tag{4.11}$$

Because of (4.8), the quantity on the left in (4.10) is equal to the dimensionless tangential stress $2D_{r\theta}$ at the wall r = 1. If the liquid fills the horizontal container completely i.e., Γ is empty, and if the attachment condition is obeyed at the wall, the sole mode of stationary flow is rotation as a solid: $v_r = 0$, $v_\theta = r$, so $D(v) = 0$. When there is a free boundary, the rotation of the liquid relative to the container is retarded, which leads to a nonzero (and nonpositive) frictional stress at the wall. This is expressed by (4.10), while (4.11) can be written as $(v_\theta - 1)D_{r\theta} = 0$ for $-\pi - \alpha < \theta < -\pi - \alpha + \delta$ and $\alpha - \delta < \theta < \alpha$, so it means that on the parts Σ_1 and Σ_3 of the cavity boundary, one has obedience either to the attachment condition or the condition for ideal sliding.

The second equation in (4.7) represents obedience to the attachment condition on Σ_2. If Σ_2 is empty (which corresponds to $\delta = \pi/2 + \alpha$), (4.4)-(4.11) has the trivial solution v = 0 and p = const. Conversely, if sets Σ_1 and Σ_2 are empty (i.e., if δ = 0), there is no solution in the class with finite Dirichlet integrals. This justifies introducing δ. The determination of it (δ is intuitively very small) lies outside the phenomenological description (see a discussion in [3, 14, 17]).

5. Solubility of (4.4)-(4.11). The (4.4)-(4.11) problem differs from that considered in section 3 for Re > 0 in that it cannot be referred to minimizing any functional, but it is found that it allows a general formulation in terms of a variational inequality. We first introduce some definitions.

For vector functions φ, ψ, and χ smooth in Ω, we define the expressions

$$a(\varphi, \psi) = \int_\Omega D\varphi : D\psi \, dx; \tag{5.1}$$

$$b(\varphi, \chi, \psi) = \int_\Omega \varphi \cdot \nabla\chi \cdot \psi \, dx. \tag{5.2}$$

The bilinear form $a(\varphi, \psi)$ and the trilinear one $b(\varphi, \chi, \psi)$ are defined also if the elements of $\varphi, \psi,$ and χ belong to a Sobolev space $H^1(\Omega)$ (for (5.2), this follows from the embedding of $H^1(\Omega)$ in $L^4(\Omega)$) . We define the space $\mathbf{H}^1(\Omega)$ as the closure in norm $\|\varphi\|_{\mathbf{H}^1(\Omega)} \equiv [a(\varphi, \varphi)]^{1/2}$ for the set of solenoidal vector functions $\varphi(x)$ smooth in Ω and that satisfy

$$\varphi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad \varphi_r = 0 \quad \text{on } \Sigma_1 \cup \Sigma_3; \tag{5.3}$$

$$\varphi = 0 \quad \text{on } \Sigma_2. \tag{5.4}$$

Space $\mathbf{H}^1(\Omega)$ is a Hilbert one; (5.1) defines the scalar product of the φ and ψ elements in it.

Functions from the $\mathbf{H}^1(\Omega)$ class are subject to the Korn inequality

$$\int_\Omega \nabla\varphi : \nabla\psi \, dx \leqslant C_3 \|\varphi\|^2_{\mathbf{H}^1(\Omega)}$$

and the Poincare-Friedricks inequality

$$\int_\Omega |\varphi|^2 \, dx \leqslant C_4 \|\varphi\|^2_{\mathbf{H}^1(\Omega)}$$

(C_3 and C_4 are positive constants). The form $b(\varphi, \chi, \psi)$ has the important features

$$b(\varphi, \psi, \psi) = 0; \tag{5.5}$$

$$b(\varphi, \chi, \psi) = -b(\varphi, \psi, \chi) \tag{5.6}$$

for any φ, ψ and χ from $\mathbf{H}^1(\Omega)$. . (5.5) follows from the representation $b(\varphi, \psi, \psi) = \int_\Omega \nabla \cdot \left(\frac{1}{2}|\psi|^2 \varphi\right) dx$ with (5.3) and (5.4). To demonstrate (5.6), it is sufficient to subtract (5.5) from $b(\varphi, \psi + \chi, \psi + \chi) = 0$.

We now construct a solenoidal vector field $\mathbf{w}(x)$ such that $w_r = 0$, $w_\theta = 1$ on Σ_2. The construction contains an element of choice, which we utilize, namely we put

$$w_r = -\frac{r^2 - 1}{2r}\frac{\partial \zeta}{\partial \theta}, \quad w_\theta = \frac{1}{2}\frac{\partial}{\partial r}[(r^2 - 1)\zeta], \tag{5.7}$$

in which ζ is a Hopf cut-off function that satisfies $\zeta \in C^\infty(\bar{\Omega})$; $\zeta = 0$ near the component Γ of the boundary of Ω, while $\zeta = 1$ on Σ_2, and for any Re > 0, one has

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{w})| \leqslant \frac{1}{2\mathrm{Re}} \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \tag{5.8}$$

no matter what $\mathbf{u} \in \mathbf{H}^1(\Omega)$. The method of constructing ζ with those properties has been given in [9].

We denote by $\eta(\theta)$ the trace of ζ on the arc Σ of the circle r = 1. It is clear that $\eta \in C_0^\infty[-\pi - \alpha, \alpha]$ and $\eta = 1$ for $\theta \in [-\pi - \alpha + \delta, \alpha - \delta]$. By K we denote the set $K = \{\varphi \in \mathbf{H}^1(\Omega) : \varphi_\theta \leqslant 1 - \eta(\theta) \text{ in } \Sigma_1 \cup \Sigma_3\}$, which is closed and convex in $\mathbf{H}^1(\Omega)$. We introduce a new unknown vector function $\mathbf{u}$ from

$$\mathbf{v} = \mathbf{u} + \mathbf{w} \tag{5.9}$$

in which $\mathbf{w}$ is a vector having components defined by (5.7). The function $\mathbf{u} \in \mathbf{H}^1(\Omega)$ will be defined as the solution to the variational inequality

$$\begin{aligned} a(\mathbf{u}, \mathbf{u} - \varphi) - \mathrm{Re}[b(\mathbf{u}, \mathbf{u} - \varphi, \mathbf{u}) + b(\mathbf{w}, \mathbf{u} - \varphi, \mathbf{u}) + \\ + b(\mathbf{u}, \mathbf{u} - \varphi, \mathbf{w})] \leqslant L(\mathbf{u} - \varphi) \, \forall \varphi \in K, \end{aligned} \tag{5.10}$$

in which L is a linear functional upon the space $\mathbf{H}^1(\Omega)$, defined by

$$L(\mathbf{u}) = -\int_{\Omega} (D(\mathbf{w}) : D(\mathbf{u}) - \mathrm{Re}\,\mathbf{w}\cdot\nabla\mathbf{u}\cdot\mathbf{w})\,dx. \tag{5.11}$$

One cannot determine whether (5.10) is soluble by referring to standard results in the theory of variational inequalities; instead, we consider the auxiliary problem

$$\begin{gathered} a(\mathbf{u}, \mathbf{u}-\boldsymbol{\varphi}) - \mathrm{Re}[b(\boldsymbol{\psi}, \mathbf{u}-\boldsymbol{\varphi}, \mathbf{u}) + b(\mathbf{w}, \mathbf{u}-\boldsymbol{\varphi}, \mathbf{u}) + \\ + b(\mathbf{u}, \mathbf{u}-\boldsymbol{\varphi}, \mathbf{w})] \leqslant L(\mathbf{u}-\boldsymbol{\varphi}) \;\; \forall \boldsymbol{\varphi} \in K \end{gathered} \tag{5.12}$$

in which ψ is a specified element in K. By $a_{\psi}(\mathbf{u}, \boldsymbol{\varphi})$ we denote a form bilinear in $\mathbf{u}$ and $\boldsymbol{\varphi}$:

$$a_{\psi}(\mathbf{u}, \boldsymbol{\varphi}) = a(\mathbf{u}, \boldsymbol{\varphi}) - \mathrm{Re}[b(\boldsymbol{\psi}, \boldsymbol{\varphi}, \mathbf{u}) + b(\mathbf{w}, \boldsymbol{\varphi}, \mathbf{u}) + b(\mathbf{u}, \boldsymbol{\varphi}, \mathbf{w})].$$

Then (5.12) is rewritten as

$$a_{\psi}(\mathbf{u}, \mathbf{u}-\boldsymbol{\varphi}) \leqslant L(\mathbf{u}-\boldsymbol{\varphi}) \;\; \forall \boldsymbol{\varphi} \in K. \tag{5.13}$$

The form $a_{\psi}(\mathbf{u}, \boldsymbol{\varphi})$ is coercive upon K − K for any $\boldsymbol{\psi} \in K$, since

$$a_{\psi}(\mathbf{u}-\boldsymbol{\varphi}, \mathbf{u}-\boldsymbol{\varphi}) \geqslant \frac{1}{2}\|\mathbf{u}-\boldsymbol{\varphi}\|^2_{\mathbf{H}^1(\Omega)} \quad \forall \mathbf{u}, \; \boldsymbol{\varphi} \in K.$$

To prove this inequality, it is sufficient to use the definition of (5.1) for form a, identity (5.5), and inequality (5.8). The Lyons–Stampacci theorem [15] implies that (5.13) will have a solution, as will (5.12), and a unique one at that, $\mathbf{u} \in K$, for any $\boldsymbol{\psi} \in K$, which defines the nonlinear operator $A: K \to K$, which puts the solution $\mathbf{u} = A(\boldsymbol{\psi})$,to (5.12 into correspondence with an element of ψ.

We demonstrate that A is continuous. Let ψ_1 and ψ_2 be any elements in K. We first put in (5.12) that $\boldsymbol{\psi} = \boldsymbol{\psi}_1$, $\mathbf{u} = \mathbf{u}_1 \equiv A(\boldsymbol{\psi}_1)$, $\boldsymbol{\varphi} = \mathbf{u}_2 \equiv A(\boldsymbol{\psi}_2)$, and then $\boldsymbol{\psi} = \boldsymbol{\psi}_2$, $\mathbf{u} = \mathbf{u}_2$, $\boldsymbol{\varphi} = \mathbf{u}_1$, and add the resulting inequalities. We use (5.1), (5.5), and (5.6) with the identity $b(\boldsymbol{\psi}_1, \mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1) + b(\boldsymbol{\psi}_2, \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_2) = b(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2, \mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1)$, to get

$$\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{H}^1(\Omega)} - \mathrm{Re}[b(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2, \mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1) + b(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2, \mathbf{w})] \leqslant 0.$$

Then from (5.8) and the Cauchy–Bunyakovskii inequality we get the bound

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{H}^1(\Omega)} \leqslant 2\mathrm{Re}\|\mathbf{u}_1\|_{\mathbf{L}^4(\Omega)}\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbf{L}^4(\Omega)}. \tag{5.14}$$

As $\mathbf{H}^1(\Omega)$ is embedded in $\mathbf{L}^4(\Omega)$, (5.14) implies that operator A is continuous.

By K_N we denote the set $K_N = \{\boldsymbol{\varphi} \in K : \|\boldsymbol{\varphi}\|_{\mathbf{H}^1(\Omega)} \leqslant N\}$ and show that for a sufficiently large N > 0, the inclusion $\mathbf{u} = A\boldsymbol{\psi} \in K_N$, applies if $\boldsymbol{\psi} \in K$. We put $\boldsymbol{\varphi} = 0$ in (5.12), which is possible by virtue of the definition of K, and from (5.5) we get

$$\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} - \mathrm{Re}\, b(\mathbf{u}, \mathbf{u}, \mathbf{w}) \leqslant L(\mathbf{u}).$$

We use the (5.8) bound with the (5.11) definition to get from the latter inequality that

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leqslant 2\left(\|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{w}\|^2_{\mathbf{L}^4(\Omega)}\right) \equiv N, \tag{5.15}$$

as was required.

Set K_N is closed and convex in $\mathbf{H}^1(\Omega)$, Let $\{\boldsymbol{\psi}_m\}$, $m = 1, 2, \ldots$,be any element sequence in K_N that converges weakly in $\mathbf{H}^1(\Omega)$. The sequence $\{\mathbf{u}_m\}$ of the corresponding solutions to (5.12) converges strongly in $\mathbf{H}^1(\Omega)$. The proof of this is based on the compactness of the embedding operator for the space $\mathbf{H}^1(\Omega)$ in $\mathbf{L}^4(\Omega)$, together with (5.14) and the (5.15) a priori bound. This implies that A is completely continuous and transfers K_N into itself (N is defined in (5.15)). Schauder's theorem shows that this operator has a fixed point in K_N, which corresponds to the solution $\mathbf{u}$ to (5.10).

From u we derive v via (5.9). By virtue of (5.3) and (5.4), which **u** satisfies as an element of $\mathbf{H}^1(\Omega)$, and by virtue of the (5.7) definition for w, conditions (4.5), (4.7), and (4.8) are obeyed for v. Condition (4.9) is met because of (5.7) and (5.9) and the definition of K. Then we use (5.10) and proceed by analogy with Sec. 3 to show readily that **v** is the general solution to (4.4) and satisfies (4.6). We now assume additionally that $\mathbf{v} \in \mathbf{C}^0(\Omega)$, and then in accordance with the Sec. 3 arguments we establish that (4.10) and (4.11) apply for v_θ.

We summarize what has been said. The (5.10) problem has a solution **u** for any Re $\geq$ 0. The function $\mathbf{v} = \mathbf{u} + \mathbf{w}$ satisfies (4.4)-(4.9) and also satisfies (4.10) and (4.11) subject to the additional assumption on the continuity in Ω.

6. Concluding Remarks. A. The Sec. 3 scheme has been used to examine the axisymmetric model problem of capillary filling, which amounts to a (3.10) variational inequality for axisymmetric vector fields.

B. The rotating-container problem allows of extension to the case where the dynamic contact angle has different values at the points where the liquid is advancing on the wall and receding from it. This contact-angle hysteresis has repeatedly been observed, as in the experiments discussed in [17]. The changes in the formulation for (4.4)-(4.11) are related only to the new definition of Ω and the components of the boundary Σ_1, Σ_2, Σ_3, and Γ.

C. Section 5 shows that a solution exists to (5.10), and it can be shown that this solution is unique for sufficiently small Re.

D. We have not considered here whether the problems are soluble for moving points of contact in an exact formulation, which are problems with unknown free boundaries. One hopes that variational inequalities will prove effective here, at least for small Ca. However, for that purpose one needs first to examine the smoothness of the solutions to (3.10) and (5.10). It has been shown [18] that the problem is soluble with an unknown boundary of (1.1)-(1.7) type, in which the attachment condition at the capillary walls is replaced by the condition for proportionality between the tangential stress and the difference between the tangential velocities of the liquid and wall.

E. There is an open question on the structure of the contact set in the solutions to (3.10) and (5.10), but some definite information can be obtained from a local analysis on how the solution behaves near the points where the attachment and slipping conditions change. We consider capillary filling for definiteness. Let one of those points have coordinates $x_1 = -\delta <$, $x_2 = 1$, and for sufficiently small $|x_1 + \delta|$, $x_1 > -\delta$, the conditions for ideal slip (2.8) and (2.9) are met, while for $x_1 < -\delta$, the (2.10) attachment conditions are met (the case where the inequalities have opposite signs can be examined similarly). We introduce the polar coordinates $r = [(x_1 + \delta)^2 + (x_2 - 1)^2]^{1/2}$, $\theta = \operatorname{arctg}[(x_2 - 1)/(x_1 + \delta)]$ and denote by v_r and v_θ the corresponding projections of the velocity vector **v**. We define the current function $\psi(r, \theta)$ by $v_r = r^{-1}\partial\psi/\partial\theta$, $v_\theta = -\partial\psi/\partial r$, and from (2.1) and (2.8)-(2.10), ψ satisfies

$$\Delta\Delta\psi = 0 \text{ for } r < \varepsilon, \ -\pi < \theta < 0 \tag{6.1}$$

($\varepsilon \in (0, \delta)$ is a certain constant) as well as the edge conditions

$$\psi = 0, \ \Delta\psi = 0 \text{ for } \theta = 0, \ 0 < r < \varepsilon; \tag{6.2}$$

$$\psi = 0, \ \partial\psi/\partial\theta = r \text{ for } \theta = -\pi, \ 0 < r \leqslant \varepsilon. \tag{6.3}$$

We additionally specify that ψ belongs to the Sobolev space H^2 in the semicircle $S_\varepsilon = \{r, \theta: r < \varepsilon, -\pi < \theta < 0\}$, which guarantees that the Dirichlet integral for vector v in region S_ε is finite.

The $r \to 0$ asymptote for the solution $\psi \in H^2(S_\varepsilon)$ to (6.1) satisfying (6.2) and (6.3) is examined by a method developed in [19, 20]. We omit the details and give the result:

$$\psi = -r\sin\theta + kr^{3/2}(\sin\theta/2 + \sin 3\theta/2) + O(r^2 \ln r) \tag{6.4}$$

for $r \to 0$, $-\pi \leqslant \theta \leqslant 0$ (k is a certain constant). The asymptotic representations for $\partial\psi/\partial r$, $\partial\psi/\partial\theta$, $\Delta\psi$ are obtained from (6.4) by formal differentiation, and in particular for $\Delta\psi$

$$\Delta\psi = 2kr^{-1/2}\sin\theta/2 + O(\ln r). \quad (6.5)$$

Then we note that by virtue of $\psi = 0$ on $0 < r \leqslant \varepsilon, \theta = -\pi$, we have $\partial v_1/\partial x_2 = \Delta\psi$ (v_1 is the projection of v on the x_1 axis). That equation together with (6.5) and (3.14) leads to the conclusion that $k \geq 0$. On the other hand, by virtue of (6.4), we have $v_1 = -1 + 2kr^{1/2} + O(r \ln r)$ for $r \to 0$ and $\theta = 0$, which agrees with (3.2) only for $k \leq 0$. Consequently, k = 0. This suggests that the solution to (3.10) has greater regularity near the points where the attachment and slip conditions change than does the solution to (2.1)-(2.10), in which the positions of those points are prescribed in advance.

Note Added in Proof. V. A. Kondrat'ev pointed out to one of us that the residual terms in (6.4) and (6.5) can be replaced by $O(r^2)$ and O(1) correspondingly, which implies that the tangential stress is bounded near the points where the attachment and slip conditions interchange. We are indebted to V. A. Kondrat'ev for that information.

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